HERMITE POLYNOMIALS

Exercise 1. Let $N_1, N_2 \sim N(0,1)$ be jointly Gaussian with $\mathbb{E}[N_1 N_2] = \rho$. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two square-integrable functions with respect to the Gaussian measure, that is, $f, g \in L^2(\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx)$. We recall that f and g can be expanded in a unique way in terms of Hermite polynomials as

$$f(x) = \sum_{k=0}^{\infty} a_k\, H_k(x) \quad \text{and} \quad g(x) = \sum_{k=0}^{\infty} b_k\, H_k(x).$$

1. Explain why $\rho \in [-1,1]$ necessarily.
2. Show that $a_0 = \mathbb{E}[f(N_1)]$.
3. Show that $\mathrm{Var}(f(N_1)) = \sum_{k=1}^{\infty} k! a_k^2$.
4. Show that $\mathrm{Cov}(f(N_1), g(N_2)) = \sum_{k=1}^{\infty} k! a_k b_k \rho^k$.
5. Deduce Gebelein's inequality: $|\mathrm{Cov}(f(N_1), g(N_2))| \le |\rho| \sqrt{\mathrm{Var}(f(N_1))\mathrm{Var}(g(N_2))}$.

Exercise 2. Let $N_1, N_2 \sim N(0,1)$ be two jointly Gaussian variables.

1. Explain why x^{2m} can be expanded in terms of Hermite polynomials as follows:

$$x^{2m} = \sum_{l=0}^{m} a_l H_{2l}(x).$$

   Here, a_l stand for some coefficients that are NOT to be determined explicitly.

2. Deduce that $\mathbb{E}[N_1^{2m} N_2^{2m}] \ge (2m-1)!!^2$, where $(2m-1)!! = (2m-1) \times (2m-3) \times \ldots \times 3 \times 1$.

From now on, we let ξ and η be two vectors of $\mathbb{R}^2$ and we write $\|\cdot\|$ (resp. $\langle \cdot, \cdot \rangle$) for the Euclidean norm (resp. the Euclidean scalar product) of $\mathbb{R}^2$. We assume further that $\|\xi\| = \|\eta\| = 1$.

3. Show that $\sup_{\|x\|=1} |\langle \xi, x\rangle| = \sup_{\|x\|=1} |\langle \eta, x\rangle| = 1$.
4. Let $X \sim N_2(0, I_2)$ and let $m \ge 1$ be an integer. Using polar coordinates, show that

$$\mathbb{E}[\langle \xi, X\rangle^{2m} \langle \eta, X\rangle^{2m}] = 4^m (2m)! \frac{1}{2\pi} \int_0^{2\pi} \langle \xi, x_\theta\rangle^{2m} \langle \eta, x_\theta\rangle^{2m} d\theta,$$

   where $x_\theta = (\cos\theta, \sin\theta) \in \mathbb{R}^2$.

5. Set $M = \sup_{\|x\|=1} |\langle \xi, x\rangle \langle \eta, x\rangle|$. Show that

$$M \ge \frac{1}{2}(2m)!^{-\frac{1}{2m}} \mathbb{E}[\langle \xi, X\rangle^{2m} \langle \eta, X\rangle^{2m}]^{\frac{1}{2m}}.$$

6. Deduce from the previous questions that

$$M \geq \frac{1}{2}(2m)!^{-\frac{1}{2m}}(2m-1)!!^{\frac{1}{m}}.$$

7. By letting $m \to \infty$, deduce finally that

$$\sup_{\|x\|=1} |\langle \xi, x\rangle\langle \eta, x\rangle| \geq \frac{1}{2}.$$

MALLIAVIN DERIVATIVE AND CHAOTIC EXPANSION

Exercise 3. Let B be a standard Brownian motion on $\mathbb{R}_+$ and let $T > 0$. For each of the following expressions for F, compute its Malliavin derivative and its chaotic expansion.

1. $F = B_T^n$ with $n \in \mathbb{N}^*$.
2. $F = e^{B_T}$.
3. $F = \int_0^T B_u du$.
4. $F = \int_0^T (B_{u+1} - B_u)^2 du$.

ABSOLUTE CONTINUITY

Exercise 4. Let B be a standard Brownian motion on $\mathbb{R}_+$. Let $x_0 \in \mathbb{R}$ and let $\sigma, b : \mathbb{R} \to \mathbb{R}$ be C^1 and (globally) Lipschitz. Consider the strong solution $X = (X_t)_{t\geq 0}$ of the stochastic differential equation (or, more correctly, stochastic integral equation):

$$X_t = x_0 + \int_0^t b(X_u)du + \int_0^t \sigma(X_u)dB_u.$$

The goal of this exercise is to show that if $\sigma(x_0) \neq 0$, then X_t has a density for any $t > 0$.

1. Let $z = (z_u)_{u\in[0,T]}$ be a simple adapted process, that is,

   $$z_u = \sum_{i=1}^{k} \xi_i \mathbf{1}_{(t_i,t_{i+1}]}(u),$$

   for some positive integer k, a finite sequence $t_0 = 0 < t_1 < \ldots < t_{k+1} = T$, and random variables $\xi_1, \ldots, \xi_k$ such that ξ_i is $\mathcal{F}_{t_i}$-measurable. Assume further that $\xi_i \in \mathbb{D}^{1,2}(\Omega)$ for each i. For any $s \in [0, T]$, show that

   $$D_s\left(\int_0^T z_u du\right) = \int_0^T D_s z_u du \quad (1)$$

   $$D_s\left(\int_0^T z_u dB_u\right) = z_s + \int_0^T D_s z_u dB_u. \quad (2)$$

   By approximation, one can show that (1)-(2) extend to any adapted process z (not necessarily simple) such that $z_u \in \mathbb{D}^{1,2}(\Omega)$ for $u \in [0, T]$ and $\int_0^T \mathbb{E}[(D_s z_u)^2]du < \infty$.

2. For any $s, t > 0$, show that $D_s X_t = 0$ if $s > t$ and

$$D_s X_t = \sigma(X_s) \exp\left\{ \int_s^t \left[b'(X_u) - \frac{1}{2}\sigma'^2(X_u) \right] du + \int_s^t \sigma'(X_u) dB_u \right\} \quad \text{if } s \leq t.$$

3. Conclude.

Exercise 5. Let $F \in \mathbb{D}^{1,2}(\Omega)$ be such that $\mathbb{E}[F] = 0$, and let us introduce the function $g_F : \mathbb{R} \to \mathbb{R}$ defined through the following identity

$$g_F(F) = \mathbb{E}\left[\langle DF, -DL^{-1}F \rangle_{L^2(\mathbb{R}_+)} \,|\, F \right].$$

1. Let C be a Borel set of $\mathbb{R}$, and set $\phi_C(x) = \int_0^x \mathbf{1}_C(t)dt$ (with the usual convention $\int_0^x = -\int_x^0$ for negative x).
   (a) Show that $x\phi_C(x) \geq 0$ for all $x \in \mathbb{R}$.
   (b) Deduce that $\mathbb{E}[g_F(F)\mathbf{1}_{\{F \in C\}}] \geq 0$.
   (c) Conclude that $g_F(F) \geq 0$ a.s.
2. If $g_F(F) > 0$ a.s., show that F has a density.
3. Assume conversely that F has a density, say ρ. Show that $g_F(F) = \frac{\int_F^\infty y\rho(y)dy}{\rho(F)}$ and deduce that $g_F(F) > 0$ a.s.

MALLIAVIN-STEIN APPROACH

Exercise 6. Let B be a standard Brownian motion on $\mathbb{R}_+$. For each $k \in \mathbb{N}$, we set

$$X_k = \int_0^\infty x_k(t) dB_t,$$

with $x_k : \mathbb{R}_+ \to \mathbb{R}$ square integrable and satisfying $\int_0^\infty x_k(t)x_l(t)dt = \rho(k-l)$, where $\rho(0) = 1$ and $\sum_{r \in \mathbb{Z}} |\rho(r)|^2 < \infty$. Finally, we also set

$$S_n = \sum_{k=0}^{n-1} \left({X_k}^2 - 1 \right), \quad n \geq 1.$$

1. Show that $X_k \sim N(0,1)$ for any k.
2. For each n, compute the chaotic expansion of S_n.
3. Find an expression of $\sigma_n^2 := \mathrm{Var}(S_n)$ in terms of ρ.
4. Compute an equivalent of $\mathrm{Var}(S_n)$ when $n \to \infty$.
5. We have $S_n = I_2(f_n)$ for some explicit $f_n \in L_s^2(\mathbb{R}_+^2)$ (see question 2). Show that $\|f_n \otimes_1 f_n\|^2_{L^2(\mathbb{R}_+^2)} \to 0$ as $n \to \infty$.
6. Deduce that $S_n/\sigma_n \to N(0,1)$ as $n \to \infty$.

Exercise 8. Let F a pth multiple Wiener-Itô integral of the form $F = I_p(f)$, with $f \in L^2_s(\mathbb{R}^p_+)$. Assume further that $\mathbb{E}[F^2] = 1$. In what follows, L denotes the generator of the Ornstein-Uhlenbeck semigroup.

1. Show that $\langle DF, -DL^{-1}F \rangle_{L^2(\mathbb{R}_+)} = \frac{1}{p} \|DF\|^2_{L^2(\mathbb{R}+)}$.

2. Set $G = F^2$. Explain why G admits a chaotic decomposition of the form

   $$(*) \qquad\qquad G = 1 + I_2(g_1) + \ldots + I_{2p}(g_p)$$

   for some $g_1, \ldots, g_p$ whose explicit expression will NOT be needed in the sequel. (By the way, I do NOT ask you to compute $g_1, \ldots, g_p$ in (*).) After expressing it by means of $\|g_k\|^2_{L^2(\mathbb{R}^{2k}_+)}$, deduce that $\mathbb{E}[LG(LG + 2pG)]$ is negative.

3. Show that

   $$LG = L(F^2) = -2pF^2 + 2\|DF\|^2_{L^2(\mathbb{R}_+)}$$

   and deduce that

   $$\mathbb{E}[\|DF\|^4_{L^2(\mathbb{R}_+)}] \leq pE[F^2\|DF\|^2_{L^2(\mathbb{R}_+)}].$$

4. Using that $F = -\frac{1}{p}LF$ and that $F^2 = F \times F$, show that

   $$1 = \mathbb{E}[F^2] = \frac{1}{p}\mathbb{E}[\|DF\|^2_{L^2(\mathbb{R}_+)}].$$

5. Using that $F = -\frac{1}{p}LF$ and that $F^4 = F^3 \times F$, show that

   $$\mathbb{E}[F^4] = \frac{3}{p}\mathbb{E}[F^2\|DF\|^2_{L^2(\mathbb{R}_+)}].$$

6. By plugging everything together, deduce that

   $$\mathbb{E}[(1 - \langle DF, -DL^{-1}F \rangle_{L^2(\mathbb{R}_+)})^2] \leq \frac{1}{3}(\mathbb{E}[F^4] - 3).$$

7. Deduce that

   $$\sup_{A \in \mathcal{B}(\mathbb{R})} \big|P(F \in A) - P(N \in A)\big| \leq \frac{2}{\sqrt{3}}\sqrt{\mathbb{E}[F^4] - 3},$$

   where $N \sim N(0,1)$ is a standard Gaussian random variable.

Exercise 9. Let B be a standard Brownian motion, let $f_n : \mathbb{R}^2_+ \to \mathbb{R}$ be symmetric and square-integrable, and let $(Z_n)_{n \geq 1}$ be the sequence defined by $Z_n = I_2(f_n)$. Assume moreover that $\mathbb{E}[Z_n^2] = 2$ for all n.

1. Express $\mathbb{E}[Z_n^2]$ in terms of f_n.

2. Show that
$$\mathbb{E}[Z_n^3] = 8\int_{\mathbb{R}_+^3} f_n(x,y)f_n(y,z)f_n(x,z)dxdydz$$
and
$$\mathbb{E}[Z_n^4] = 12 + 48\int_{\mathbb{R}_+^2} f_n(x,y)^2 dxdy.$$

3. Deduce that
$$\begin{aligned} & 48\int_{\mathbb{R}_+^2}\left[f_n(x,y) - \int_0^\infty f_n(x,z)f_n(y,z)dz\right]^2 dxdy \\ = \; & 48 - 12\mathbb{E}[Z_n^3] + 48\int_{\mathbb{R}_+^2}\left(\int_0^\infty f_n(x,z)f_n(y,z)dz\right)^2 dxdy. \end{aligned}$$

4. Show that
$$\begin{aligned} & \mathbb{E}\big[(\|DZ_n\|_{L^2(\mathbb{R}_+)}^2 - 4Z_n - 4)^2\big] \\ = \; & 32\int_{\mathbb{R}_+^2}\left[f_n(x,y) - \int_0^\infty f_n(x,z)f_n(y,z)dz\right]^2 dxdy. \end{aligned}$$

5. As $n \to \infty$, deduce that $\|DZ_n\|_{L^2(\mathbb{R}_+)}^2 - 4Z_n \to 0$ in $L^2(\Omega)$ if and only if $\mathbb{E}[Z_n^4] - 12\mathbb{E}[Z_n^3] \to -36$.